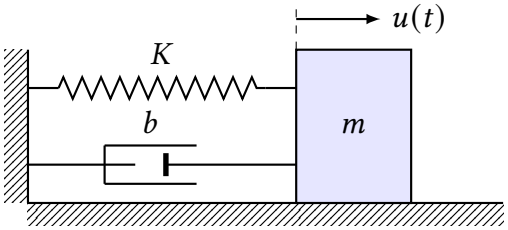
Q1. Damped Harmonic Oscillator.

A mass m is attached to a wall by a spring of stiffness K and a damper with damping constant b. Let $u(t)$ be the displacement of the mass from rest, and $v(t) = u'(t)$ its velocity. The spring pulls back with force $-Ku$ and the damper resists motion with force $-bu'$.



(a) Use Newton's second law to write the motion of the mass as an ODE in u. Then write it as a first order system for $\mathbf{y} = (u, v)^\top$, i.e. $\mathbf{y}' = A\mathbf{y}$, and identify A.
Hint: write the ODE in terms of the natural frequency ω_0 and the damping rate σ_0, defined by $\omega_0^2 = K/m$ and $2\sigma_0 = b/m$. Use ω_0 and σ_0 in (b) and (c) as well.

(b) Let $\mathbf{y}_k = (u_k, v_k)^\top \approx \mathbf{y}(t_k)$ with $t_k = kh$. Write one step of **forward Euler** as a recursion for $\mathbf{y}_{k+1}$ in terms of $\mathbf{y}_k$.

(c) Do the same with **backward Euler**. Rearrange the recursion so that only $\mathbf{y}_{k+1}$ appears on the left side, i.e. $\mathbf{y}_{k+1} = B\,\mathbf{y}_k$, and find the matrix B.

Q2. A Nonlinear ODE.

Consider the scalar initial value problem

$$u'(t) = -u(t)^2, \qquad u(0) = u_0 > 0,$$

whose exact solution is $u(t) = u_0/(1 + u_0 t)$. Let $u_k \approx u(t_k)$ with $t_k = kh$.

(a) Write one step of **forward Euler** as a recursion for u_{k+1}.

(b) Write one step of **backward Euler** as a recursion for u_{k+1}.

(c) Now consider $u'(t) = -\sin u(t)$ instead. Write one step of **backward Euler**. Can you rearrange the recursion so that only u_{k+1} is on the left side? If not, how can we compute u_{k+1} from u_k?