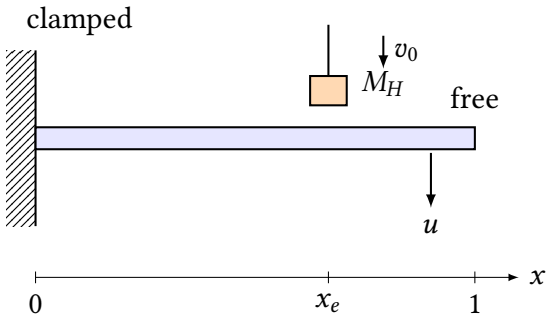
Setup (used in both problems).

A uniform bar of length L is **clamped** at $X = 0$ and **free** at $X = L$ (a tuning fork tine or a kalimba tine). It has Young's modulus E, area moment I, density ρ and cross section A, so its mass is $M_b = \rho AL$. Let $x = X/L \in [0, 1]$ and let $u(x, t)$ be the transverse displacement, measured positive **downward**. A hammer of mass M_H and position $u_H(t)$ (also downward) drops onto the bar from above at x_e. It starts just touching, $u_H(0) = 0$, moving down with $u_H'(0) = v_0 > 0$, and the bar starts at rest. While they touch, the felt is compressed by $u_H - u(x_e, t)$ and pushes the bar and the hammer apart with force

$$f(t) = K_H \left[u_H(t) - u(x_e, t)\right]_+^\alpha, \qquad M_H\, u_H'' = -f.$$



Gravity is neglected. The PDE is linear, so the bar's weight only adds a static sag, and we measure u from that sagged rest position. During a contact of about 1 ms, gravity changes the hammer's speed by only $g \cdot 1\,\text{ms} \approx 1\,\text{cm/s}$.

Q1. The Struck Cantilever: Finite Differences.

(a) Write the PDE for $u(x, t)$ on $0 < x < 1$, including a loss term $-2\sigma_0 u_t$ and the hammer force applied at the point x_e. Then write the boundary conditions at $x = 0$ and $x = 1$ and the initial conditions.
Hint: the bending stiffness gives $\rho A\, u_{tt} = -EI\, u_{XXXX}$ before scaling. Use $\kappa^2 = EI/(\rho AL^4)$ as in the lecture, and write the point force with a Dirac delta, keeping in mind that $\delta(X - X_e) = \delta(x - x_e)/L$.

(b) Use the grid $x_l = lh$, $h = 1/N$, $t_n = nk$, and $u_l^n \approx u(x_l, t_n)$. Discretize the PDE with δ_{tt}, $\delta_{xxxx} = \delta_{xx}\delta_{xx}$ and $\delta_{t\cdot}$ for the loss. Spread the force onto the grid point l_e nearest to x_e, and read the bar there. Write the explicit update for u_l^{n+1}, and the update for the hammer u_H^{n+1}.

(c) Impose the boundary conditions with virtual grid points: u_{-1} at the clamped end, and u_{N+1}, u_{N+2} at the free end. Write $h^4\delta_{xxxx}u$ at $l = 1,\ 2,\ N-1,\ N$ in terms of the unknowns $u_1, \ldots, u_N$ only, and show the matrix $\mathbf{D}$ with $h^4\delta_{xxxx}\mathbf{u} = \mathbf{D}\mathbf{u}$ for $N = 6$.

(d) The scheme is stable if $\mu = \kappa k/h^2 \le \frac{1}{2}$. Take $\kappa = 882\ \text{s}^{-1}$ (a bar whose lowest mode is near 494 Hz, B4). At $f_s = 44.1$ kHz, what is the largest N you can use? What do you expect for the pitch of the simulated bar, and why?

Q2. The Struck Cantilever: Modal Synthesis. Solve the same PDE from Problem 1(a) by expanding u in the modes of the clamped-free bar,

$$u(x,t) = \sum_{p=1}^{P} \phi_p(x)\, q_p(t).$$

(a) Find the mode shapes $\phi_p(x)$ and the frequencies ω_p. Show that the wavenumbers β_p satisfy

$$\cos\beta\, \cosh\beta = -1,$$

This has no closed form solution. Explain why $\beta_p \approx (2p-1)\pi/2$ for large p, then refine $\beta_1, \beta_2, \beta_3$ with Newton's method (class 5) starting from that guess. Give the ratios ω_p/ω_1 for $p = 2, 3$.

(b) Insert the expansion into the forced, lossy PDE and derive one ODE for each $q_p(t)$. How does the hammer enter each mode? How do you compute $u(x_e, t)$, which the hammer force needs?

(c) Write a time stepping recursion for $q_p^n \approx q_p(t_n)$ that has **exact** pitch, together with the hammer update. Compare with Problem 1: what limits k and h now, and what is the only approximation?

Hint: how to solve a PDE in modal form.

1. **Separate variables.** Drop the force and loss, and insert $u = \phi(x)\, e^{j\omega t}$ into $u_{tt} = -\kappa^2 u_{xxxx}$. You get an ODE in x only, $\phi'''' = \beta^4\phi$ with $\omega = \kappa\beta^2$.

2. **General solution.** $\phi = a\cosh\beta x + b\sinh\beta x + c\cos\beta x + d\sin\beta x$. Each boundary condition is a linear equation in (a, b, c, d). The two at $x = 0$ remove two unknowns; the two at $x = 1$ give a 2×2 homogeneous system.

3. **Eigenvalue condition.** A nonzero ϕ exists only if the determinant of that system is zero. This equation in β has infinitely many roots β_p, and each gives a mode ϕ_p and a frequency ω_p. To locate them, note that $\cosh\beta$ grows very fast.

4. **Orthogonality.** Modes of the same operator with these boundary conditions are orthogonal. Scale them so that $\int_0^1 \phi_p\phi_q\, dx = \delta_{pq}$ (for the form in step 2, this happens with $a = 1$).

5. **Project.** Insert $u = \sum_q \phi_q q_q$ into the full PDE, multiply by $\phi_p(x)$ and integrate over $[0, 1]$. Orthogonality keeps only the p-th term; $\int \phi_p\, \delta(x - x_e)\, dx = \phi_p(x_e)$.

6. **Time step.** Each mode is a damped harmonic oscillator, so use the exact recursion from class 13 (the same as for the marimba bar and the modal plate reverb in class 15).